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Correspondence between quantum Heisenberg models (spin-$\frac{1}{2}$) and bosonic models

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Abstract. It is shown that the partition function of a quantum Heisenberg model of spin-$\frac{1}{2}$ can be written as the partition function of a Yukawa-Euclidean quantum field theory, thus making the statistical mechanical problem amenable to analysis through techniques of constructive field theory.

1. Introduction

Quantum spin systems such as the Heisenberg model have been the subject of intensive theoretical research for a long time [1]. In spite of a large number of different approaches to these problems and the several rigorous results known [2–4], there are still many interesting and difficult questions concerning not only the rigorous aspect of the problems but also their physical interpretation (e.g. strongly correlated models, antiferromagnetic quantum Heisenberg).

In this paper we prove the equivalence of a large class of quantum Heisenberg models of spin-$\frac{1}{2}$ (including ferro and antiferromagnetic interactions) and the class of Euclidean field theory models known as Yukawa models. Essentially, a functional integral method is used to establish this correspondence [5].

We have thus rigorously connected the statistical mechanical problem with an area exhaustively studied in the last 25 years, namely, the constructive field theory. In this way, we may apply in principle the highly successful techniques developed in mathematical field theory to the original problem.

We should mention here that in [6] and [7] the authors also relate the quantum Heisenberg model to an Euclidean field theory (via a Feynman–Kac formula), but their approach is completely different and the resulting Euclidean field theoretic Lagrangian contains Yukawa and Luttinger type interactions.

2. Definitions and results

We are interested in the class of quantum spin systems described by Heisenberg models with Hamiltonians given by

$$\mathscr{H} = -\tfrac{1}{2}\sum S_i(\mathscr{J}_{ij} + m\delta_{ij})\cdot S_j \tag{1}$$

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where

- the sum above is taken over the sites ($i, j = 1, \ldots, N$) of a finite and d-dimensional lattice $\Lambda \subset \mathbb{Z}^d$;
- the quantum spin operators S_j are proportional to the Pauli matrices: $S_j^\alpha = \frac{1}{2}\sigma_j^\alpha$, $\alpha \in \{x, y, z\}$;
- $\mathscr{H}$ is defined on the Hilbert space

$$H \equiv \bigotimes_{i=1}^{N} H_i$$

  where $H_i \equiv \mathbb{C}^2$;
- $\mathscr{J}_{ij}$ is a decaying interaction (although this is not necessary in a finite volume)

$$\max_i \sum_j |\mathscr{J}_{ij}| < K \tag{2}$$

  (K is a positive constant);
- m is large enough (see appendix).

The partition function is defined by

$$Z = \mathrm{Tr}_H\, \mathrm{e}^{-\beta\mathscr{H}} \tag{3}$$

where β is the inverse of temperature, and Tr_H is the trace over H.

Let us remark that the mass m was introduced in (1) just to include ferro and antiferromagnetic interactions in our future formulae. The sole effect of this term m is the multiplication of $Z_{m=0}$ by a constant: $\exp[\frac{3}{8}\beta N m]$.

Following the 'hint' given by the Gaussian identity [5], we write the partition function in an integral form where the integrand depends on S_i, not $S_i \cdot S_j$, and on commutable variables. Then, after standard manipulation and some analysis, we prove the identity between Z and the partition function of a Yukawa model (represented by a pure bosonic integration [8, 9]), i.e. we obtain an expression similar to

$$Z = \int \mathrm{d}\mu(\phi)\, \det(1 + \mathscr{K}(\beta, \phi)) \tag{4}$$

where ϕ is a bosonic field defined on N sites; $\mathrm{d}\mu$ a specific measure; $\mathscr{K}$ an operator; and det is the infinite determinant [10, 11]. That is, we have a suitable formulation for the renormalization group approach to further studies in the infinite volume. Rigorously, we have a meaningful expression considering a proper and natural cutoff reguralizing the measure $\mathrm{d}\mu$. For details see the formulae (19) and (26). Also shown is the convergence of the perturbative β series for Z with this reguralized determinant, and that each term (coefficient of β^n) is finite in the limit corresponding to eliminating the regularizer.

3. The correspondence

To apply the Gaussian identity to expressions with non-commuting variables we are guided, in some sense, by the Lie product theorem which states

$$\exp[A + B] = \lim_{p \to \infty} (\exp[A/p]\exp[B/p])^p. \tag{5}$$

Actually, we have the following result, proved in the appendix:

Lemma 1

$$Z = \mathrm{Tr}_H\, e^{-\beta\mathscr{H}} = \lim_{p\to\infty} \mathrm{Tr}_H \left[\int \frac{\Pi_{k=1}^N \mathrm{d}\phi_k \exp\{\beta/p[-\frac{1}{2}(\phi_i, J_{ij}^{-1}\phi_j) + (\phi_i, S_i)]\}}{(2\pi p/\beta)^{3N/2}(\det J_{ij})^{3/2}}\right]^p \tag{6}$$

where $\phi_i \in \mathbb{R}^3$; the sum over repeated indices is considered; and $J_{ij} = \mathscr{J}_{ij} + m\delta_{ij}$.

Using this lemma we can write Z as

$$Z = \lim_{p\to\infty} \int \left(\prod_{t=\beta/p}^{\beta} \frac{\Pi_{k=1}^N \mathrm{d}\psi_k(t)}{\mathscr{N}}\right) \times \exp\left(-\frac{1}{2}\sum_{t=\beta/p}^{\beta} \frac{\beta}{p}\psi_i(t)\cdot J_{ij}^{-1}\psi_j(t)\right) \prod_{k=1}^{N} \mathrm{tr}\, T\exp\left(-\sum_{t=\beta/p}^{\beta}\frac{\beta}{p}\psi_k(t)\cdot S\right) \tag{7}$$

where tr is the trace over H_i; $\mathscr{N}$ is the normalization factor; t takes p values ($\beta/p, 2\beta/p, \ldots, \beta$); and T is the 'time ordering' due to the non-commutativity of the S matrices:

$$T\exp\left(-\sum_{t=\beta/p}^{\beta}\frac{\beta}{p}\psi_i(t)\cdot S\right) \equiv \exp\left[-\frac{\beta}{p}\psi_i(\beta)\cdot S\right]\exp\left[-\frac{\beta}{p}\psi_i\left(\frac{p-1}{p}\beta\right)\cdot S\right]\ldots\exp\left[-\frac{\beta}{p}\psi_i(\beta/p)\cdot S\right]. \tag{8}$$

Now we define the function $\phi_i(t)$ for t in the interval $[0, \beta]$ by

$$\phi_i(t) = \psi_i(n\beta/p) \qquad \text{if } t \in ((n-1)\beta/p, n\beta/p] \tag{9}$$

(and $\phi_i(0) = \psi_i(\beta/p)$), in order to write the partition function as

$$Z = \lim_{p\to\infty} \int \mathrm{d}\mu_{c_p(t,t')J_{ij}}(\phi) \prod_{k=1}^{N} \mathrm{tr}\, T\exp\left[-\int_0^\beta \mathrm{d}t\, \phi_k(t)\cdot S\right] \tag{10}$$

where ϕ is given by $(\phi_1, \ldots, \phi_N)$; the measure is Gaussian with covariance $c_p(t, t')J_{ij}$

$$c_p(t, t') = \sum_{i=1}^{p} \frac{p}{\beta}[\chi_i(t)\chi_i(t')] \tag{11}$$

and χ_i is the characteristic function of $((i-1)\beta/p, i\beta/p]$ (for $i = 1$ take $[0, \beta/p]$).

After a simple rescaling $t \to t/\beta$:

$$Z = \lim_{p\to\infty} \int \mathrm{d}\mu_{c_p(t,t')J_{ij}}(\phi) \prod_{k=1}^{N} \mathrm{tr}\, T\exp\left[-\beta\int_0^1 \mathrm{d}t\, \phi_k(t)\cdot S\right] \tag{12}$$

(now, $t \in [0, 1]$; χ_i is defined on intervals $((i-1)/p, i/p]$, and so on).

We control the time ordering in the expression above by standard procedures: introducing particle creation and anihilation operators to represent the spin matrices and using the Wick theorem. A careful analysis guides us to the following result:

Lemma 2

$$\mathrm{tr}\, T\exp\left(-\beta\int_0^1 \mathrm{d}t\, \phi(t)\cdot S\right) = 2\,\mathrm{det}_2^2[1 + \beta A(\phi)] \tag{13}$$

where $A(\phi)$ has the kernel

$$[A(\phi)]_{\lambda\mu}(t,t')=\tfrac{1}{2}\varepsilon(t,t')\phi(t')\cdot S(\lambda,\mu) \tag{14}$$

$\varepsilon(t,t')\equiv\varepsilon(t'-t)$, $\varepsilon(x)=1$ for $x>0$, or -1 for $x<0$. And $A(\phi)$ is well defined on the Hilbert space $\mathbb{H}$ given by

$$\mathbb{H}=\{f\,|\,f\in L^2_{\mathrm{AP}}[0,1]\oplus L^2_{\mathrm{AP}}[0,1]\} \tag{15}$$

with $(f,f)_{\mathbb{H}}=(f^1,f^1)+(f^2,f^2)$, where $(f^j,f^j)=\Sigma_{k\,\mathrm{odd}}|k\pi||f^j_k|^2$; AP means anti-periodic conditions; and $f_k\equiv\int_0^1 \mathrm{d}t\,f(t)\,\mathrm{e}^{\mathrm{i}\pi kt}$.

Proof. First of all we note that

$$\mathscr{g}(\beta)\equiv\operatorname{tr}T\exp\left(-\beta\int_0^1\mathrm{d}t\,\phi(t)\cdot S\right)=\operatorname{tr}T\exp\left(-\beta\sum_{k=1}^{p}\frac{1}{p}\phi(k/p)\cdot S\right)$$

and $\det_2[1+\beta A(\phi)]$ (provided A is Hilbert–Schmidt) are analytic functions in β [10, 11]. Then, it is sufficient to prove the statement for small β.

We take $\log[\mathscr{g}(\beta)]$ (it makes sense for small β since $\mathscr{g}(0)=2$) and write the spin matrices introducing anihilation and creation operators (as in [5]):

$$\log[\mathscr{g}(\beta)]=\log\Bigg\{\mathrm{Tr}_1\sum_{m=0}^{\infty}\frac{(-\beta)^m}{m!}T\int_0^1\mathrm{d}t_1\ldots\mathrm{d}t_m \\ \times\phi(t_1)\cdot S(\lambda_1,\mu_1)a^\dagger_{\lambda_1}a_{\mu_1}\ldots\phi(t_m)\cdot S(\lambda_m,\mu_m)a^\dagger_{\lambda_m}a_{\mu_m}\Bigg\} \tag{16}$$

- where μ and λ take the values $\{1,2\}$;
- $S^\gamma(\lambda,\mu)$ is the spin matrix element;
- $a^\dagger_\lambda$, a_μ are fermion creation, annihilation operators;
- Tr_1 means the trace over the space of one fermion.

After applying the Wick theorem and summing up the Feynman diagrams:

$$\mathscr{g}(\beta)=2\exp\left[2\sum_{m=2}^{\infty}\frac{(-\beta)^m}{m}\operatorname{Tr}A^m(\phi)\right]=2\left(\exp\left[\sum_{m=2}^{\infty}\frac{(-\beta)^m}{m}\operatorname{Tr}A^m(\phi)\right]\right)^2 \tag{17}$$

where $A(\phi)$ has the kernel given by (14), and Tr is understood as the sum over indices λ, μ, and the integration on t.

Defining a Hilbert space where $A(\phi)$ is Hilbert–Schmidt, the expression for $\mathscr{g}(\beta)$ becomes exactly the Plemelj–Smithies formula for $2\det_2^2[1+\beta A(\phi)]$ [10, 11]. In fact, since $\operatorname{Tr}A(\phi)=0$, $\det_2[1+\beta A(\phi)]=\det[1+\beta A(\phi)]$.

To construct the Hilbert space we consider only anti-periodic functions $f\in L^2_{\mathrm{AP}}[0,1]\oplus L^2_{\mathrm{AP}}[0,1]$ since $\varepsilon(1,t)=-\varepsilon(0,t)$. Note that the operator $\frac{1}{2}\varepsilon$ is the inverse of the derivative operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\varepsilon(t,t')=\delta(t,t').$$

Taking the basis for $L^2_{\mathrm{AP}}[0,1]$ given by $\{\exp[\lambda_k t]|\lambda_k=\pi ki,\ k\in\mathbb{Z}\text{ odd}\}$, we see that

$$\tfrac{1}{2}\varepsilon(\exp[\lambda_k t])=\frac{1}{\pi ki}\exp[\lambda_k t].$$

Writing $A(\phi)\equiv\lambda^{-1}\phi\cdot S$ (where $\lambda^{-1}\equiv\frac{1}{2}\varepsilon$), we have

$$\operatorname{Tr}[A^\dagger(\phi)A(\phi)] \\ =\mathrm{Tr}_s\sum_{k,j\,\mathrm{odd}}\left\{\frac{1}{|\lambda_j|}\frac{1}{|\lambda_k|}\int\mathrm{d}t'\,\mathrm{d}t\,\phi(t')\cdot S\phi(t)\cdot S\exp[(\lambda_k-\lambda_j)(t-t')]\right\} \tag{18}$$

where Tr_s means the trace over the S matrices space. Since (18) is finite for ϕ constant on p-intervals, it is proved that $A(\phi)$ is Hilbert–Schmidt and $\det_2[1+\beta A(\phi)]$ is well defined (notice that $\text{Tr}[A^\dagger(\phi)A(\phi)]$ diverges logarithmically as p goes to infinity). □

Now we are ready to establish the correspondence between the Heisenberg and Yukawa models.

Theorem

$$Z=\lim_{p\to\infty}\int \mathrm{d}\mu_{c_p(t,t')J_{ij}}(\phi)\prod_{k=1}^{N} 2\det{}_2^2[1+\beta A(\phi_k)] \tag{19}$$

where the power series in β is absolutely convergent and each term (coefficient of β^n) has a finite limit as $p\to\infty$.

Proof. The formula above is obvious after the lemmas. The proof for the statement about the power series in β becomes simpler if we consider the previous expression for the determinant:

$$2\det{}_2^2[1+\beta A(\phi)]=\text{tr}\, T\exp\left(-\beta\int_0^1 \mathrm{d}t\,\phi(t)\cdot S\right)$$

$$=\sum_{m=0}^{\infty}\frac{(-\beta)^m}{m!}\text{tr}\, T\int_0^1 \mathrm{d}t_1\ldots\mathrm{d}t_m\,\phi^{\alpha_1}(t_1)\ldots\phi^{\alpha_m}(t_m)S^{\alpha_1}\ldots S^{\alpha_m}$$

where the indices i (and the sum $\Sigma_{\alpha_1,\ldots\alpha_m}$) were not written in order to simplify the notation.

Let us comment that, for a collection of S matrices, $\text{tr}\, S^{\alpha_1}\ldots S^{\alpha_m}$ equals $(\pm 2)/2^m$ or 0 if m is even, and $(\pm 2\mathrm{i})/2^m$ or 0 if m is odd; and that the time ordering above affects just the sign of $\text{tr}\, S^{\alpha_1}\ldots S^{\alpha_m}$, since $\phi^{\alpha_i}(t_i)$ and $\phi^{\alpha_j}(t_j)$ commute. So we write $2\det_2^2[1+\beta A(\phi_i)]$ as

$$\sum_{m=0}^{\infty}\frac{(\beta)^m}{m!}2/2^m(\mathrm{i})^m\int_0^1 \mathrm{d}t_1\ldots\mathrm{d}t_m\, g_m^{\alpha}(t_1,\ldots t_m)\phi^{\alpha_1}(t_1)\ldots\phi^{\alpha_m}(t_m) \tag{20}$$

where $\alpha\equiv(\alpha_1,\ldots\alpha_m)$, and $g_m^\alpha(t_1,\ldots t_m)$ takes the values ± 1 (the hypercube $[0,1]^m$ is divided in sectors where g^α is constant and equals 1 or -1 according to the ordering of $(t_1,\ldots t_m)$; e.g. $t_1<\ldots<t_m$, one sector; $t_m<\ldots<t_1$, another sector; …).

Thus,

$$Z\leqslant\lim_{p\to\infty}\int \mathrm{d}\mu_{c_p(t,t')J_{ij}}(\phi)\sum_{m_1\ldots m_N}\frac{(\beta)^{m_1}}{m_1!}\cdots\frac{(\beta)^{m_N}}{m_N!}2^N/(2^{m_1+\ldots+m_N})$$

$$\times\sum_{\alpha_1^1\ldots\alpha_{m_N}^N}\left\{\left|\int_0^1 \mathrm{d}t_1^1\ldots\mathrm{d}t_{m_1}^1\, g^{\alpha^1}(t_1^1,\ldots t_{m_1}^1)\phi^{\alpha_1^1}(t_1^1)\ldots\phi_1^{\alpha_{m_1}^1}(t_{m_1}^1)\right|\ldots\right.$$

$$\left.\times\left|\int_0^1 \mathrm{d}t_1^N\ldots\mathrm{d}t_{m_N}^N\, g^{\alpha^N}(t_1^N,\ldots t_{m_N}^N)\phi^{\alpha_1^N}(t_1^N)\ldots\phi_1^{\alpha_{m_N}^N}(t_{m_N}^N)\right|\right\}.$$

Permuting the sum and the integral, and using the Schwarz inequality:

$$Z \leqslant \lim_{p\to\infty} \sum_{m_1\ldots m_N} \sum_{\alpha_1^1\ldots\alpha_{m_N}^N} \frac{(\beta)^{m_1}}{m_1!}\cdots\frac{(\beta)^{m_N}}{m_N!} 2^N/(2^{m_1+\ldots+m_N})$$
$$\times\left\{\int \mathrm{d}\mu_{c_p(t,t')J_{ij}}(\phi)\left[\left(\int_0^1 \mathrm{d}t_1^1\ldots \mathrm{d}t_{m_1}^1 \text{——}\right)\right.\right.$$
$$\left.\left.\times\ldots\left(\int_0^1 \mathrm{d}t_1^N\ldots \mathrm{d}t_{m_N}^N \text{——}\right)\right]^2\right\}^{1/2}.$$

Calculating the Gaussian integral, we obtain a bound for Z uniform on p:

$$Z \leqslant \sum_{m_1\ldots m_N} \frac{(\beta)^{m_1}}{m_1!}\cdots\frac{(\beta)^{m_N}}{m_N!} 2^N 2^{-(m_1+\ldots+m_N)} 3^{m_1+\ldots+m_N}$$
$$\times\{(J/\beta)^{m_1+\ldots+m_N}[2(m_1+\ldots+m_N)-1]!!\times 1\}^{1/2} \tag{21}$$

where

- $3^{m_1+\ldots+m_N}$ is the number of terms in
$$\sum_{\alpha_1^1\ldots\alpha_{m_N}^N}$$
- J is a bound for J_{ij} (take $J=\max_{ij}|J_{ij}|$);
- $(2(m_1+\ldots+m_N)-1)!!$ is the bound for the number of possible contractions among the ϕ^α terms (i.e. for the number of graphics);
- 1 is the bound for the g function integrations.

Thus

$$Z \leqslant \sum_{m_1\ldots m_N} 2^N (9J\beta/2)^{(m_1+\ldots+m_N)/2} \frac{1}{m_1!\ldots m_N!}[(m_1+\ldots+m_N)!]^{1/2}. \tag{22}$$

And with the proposition

$$\frac{(m_1+\ldots+m_N)!}{m_1!\ldots m_N!} \leqslant \exp[C_N(m_1+\ldots+m_N)] \tag{23}$$

(C_N depends on N) we have

$$Z \leqslant \sum_{m_1\ldots m_N} 2^N (9J\beta/2)^{(m_1/2+\ldots+m_N/2)} \frac{\exp[C_N(m_1/2+\ldots+m_N/2)]}{(m_1!\ldots m_N!)^{1/2}}$$
$$\leqslant 2^N\{[1+(\tfrac{1}{2}9J\beta\exp[C_N])^{1/2}]\exp(\tfrac{1}{2}9J\beta\exp[C_N])\}^N \tag{24}$$

where we have used that $(m_i!)^{1/2} \geqslant (\frac{1}{2}m_i)!$ and that

$$\sum \frac{X^{n/2}}{(n/2)!} \leqslant \mathrm{e}^X + X^{1/2}\,\mathrm{e}^X.$$

To prove the proposition (23), we use the Stirling formula:

$$\log\left\{\frac{(m_1+\ldots+m_N)!}{m_1!\ldots m_N!}\right\}$$
$$\simeq (m_1+\ldots+m_N)\log(m_1+\ldots+m_N)$$
$$-(m_1\log[m_1]+\ldots+m_N\log[m_N]) \tag{25}$$

and consider the function

$$f(x_1, \ldots, x_N) = (x_1 + \ldots + x_N) \log(x_1 + \ldots + x_N) - (x_1 \log[x_1] + \ldots + x_N \log[x_N])$$

where $1 \leq x_i \in \mathbb{R}$. Since $x \cdot \nabla(\partial f/\partial x_i) = 0$, we have $\partial f/\partial x_i$ constant in the radial direction and thus, $f(x_1, \ldots, x_N) \leq (x_1 + \ldots + x_N) C_N$ for $C_N = \log[N]$; which proves the proposition and the theorem. □

Remarks. The bound (constant)$^{N^2}$ obtained for Z (instead of (constant)^N) does not lead to a finite free energy density as N goes to infinity. This is due to the fact that we did not exploit the decay of J_{ij} (we used the bound $J = \max_{ij}|J_{ij}|$).

The formula

$$Z = \int d\mu_{(\delta_{(t,t')} J_{ij}/\beta)}(\phi) \prod_{k=1}^{N} 2 \det_2^2[1 + \beta A(\phi_k)] \tag{26}$$

has a formal sense if we write the Gaussian integral as a sum of Feynman graphics. Some of the graphics will not be well defined in principle (they will contain expressions with $\varepsilon(x)$ at $x = 0$), but if we follow the calculations made with the similar formula with covariance c_p instead of δ, we see that these Feynman graphics (depending on p) will coincide exactly in the limit $p = \infty$ with the graphics due to the formula above considering $\varepsilon(0) = 0$. Of course, this procedure has just a formal meaning since $\det_2$ is not rigorously defined for ϕ on the support of the white-noise measure.

We can also obtain a pure fermonic expression for Z after writing $\det_2$ in terms of fermionic variables and integrating out the bosonic field ϕ (it gives a Gross–Neveu model).

Appendix

Proof of lemma 1. First of all, we take m larger than K (K from (4)). Thus

$$(\mathscr{J} + m)^{-1} = \left(1 - \frac{\mathscr{J}/m}{\mathscr{J}/m + 1}\right) m^{-1} \tag{27}$$

is well defined and positive since $\|\mathscr{J}\| < m$ [12]. We emphasize that the statement is valid even for antiferromagnetic interactions.

Now we calculate the expression

$$\lim_{p \to \infty} \left[\int \frac{\prod_{k=1}^{N} d\phi_k \exp\{-\frac{1}{2}(\phi_i, J_{ij}^{-1}\phi_j) + \lambda(\phi_i, S_i)\}}{(2\pi)^{3N/2} (\det J_{ij})^{3/2}} \right]^p \tag{28}$$

(where $\lambda \equiv (\beta/p)^{1/2}$) writing $\exp[\lambda(\phi_i, S_i)]$ as a power series in λ and integrating each term λ^n. After the integration, reordering properly the S_i^α matrices, we have (for the term order $2n$):

$$[(2n-1)!!(S_i, J_{ij} S_j) + Y(n)] \lambda^{2n}/(2n)! \tag{29}$$

where $Y(n)$ contains the extra terms generated by reordering the non-commuting matrices ($[S_k^\alpha, S_j^\beta] = i\delta_{kj}\varepsilon^{\alpha\beta\gamma} S_j^\gamma$).

To understand $Y(n)$, let us consider the term with four points as an example:

$$\int d\mu_J(\phi) \phi_{i_1} \cdot S_{i_1} \phi_{i_2} \cdot S_{i_2} \phi_{i_3} \cdot S_{i_3} \phi_{i_4} \cdot S_{i_4}.$$

After the integration we have

$$J_{i_1 i_2} J_{i_3 i_4} S^\alpha_{i_1} S^\alpha_{i_2} S^\beta_{i_3} S^\beta_{i_4} + J_{i_1 i_3} J_{i_2 i_4} S^\alpha_{i_1} S^\beta_{i_2} S^\alpha_{i_3} S^\beta_{i_4} + J_{i_1 i_4} J_{i_2 i_3} S^\alpha_{i_1} S^\beta_{i_2} S^\beta_{i_3} S^\alpha_{i_4}$$

where $\alpha, \beta \in \{x, y, z\}$, $i_k \in \{1, \ldots, N\}$ and the sum over the indices is considered.

The second term above, if reordered, equals

$$J_{i_1 i_3} J_{i_2 i_4} S^\alpha_{i_1} S^\alpha_{i_3} S^\beta_{i_2} S^\beta_{i_4} + J_{i_1 i_3} J_{i_2 i_4} S^\alpha_{i_1} [S^\beta_{i_2}, S^\alpha_{i_3}] S^\beta_{i_4}$$

generating one extra term which is considered in $Y(4)$.

For the order $2n$, there are $(2n-1)!!$ terms due to the integration of $\phi_{i_1} \ldots \phi_{i_{2n}}$. To evaluate the number of extra terms in $Y(n)$ generated by reordering these $(2n-1)!!$ factors, we note that the term which requires more commutations to be adjusted is

$$J_{i_1 i_{2n}} J_{i_2 i_{2n-1}} J_{i_3 i_{2n-2}} \ldots S^\alpha_{i_1} S^\beta_{i_2} S^\gamma_{i_3} \ldots S^\gamma_{i_{2n-2}} S^\beta_{i_{2n-1}} S^\alpha_{i_{2n}}.$$

Adjusting this, i.e., writing it as $S^\alpha_{i_1} S^\alpha_{i_{2n}} S^\beta_{i_2} S^\beta_{i_{2n-1}} \ldots$, we generate $n(n-1)$ new factors. Consequently, the number of extra terms for the order $2n$ is bounded by $(2n-1)!!\,n(n-1)$.

Considering all orders we have

$$\sum_{n=2}^{\infty} \frac{\lambda^{2n}}{(2n)!} Y(n) = (\lambda^2/2)^2 \sum_{n=2}^{\infty} \frac{(\lambda^2/2)^{2n-2}}{n!(2n-1)!!} Y(n) \equiv (\lambda^2/2)^2 \mathscr{S} \tag{30}$$

where the sum makes sense (the number of terms in $Y(n)$ is not large enough to spoil the sum). Thus, using this result we write the expression (28) as

$$\lim_{p\to\infty} [\exp[(S_i, J_{ij} S_j)\beta/2p] + \mathscr{S}(\beta^2/4p^2)]^p = \exp[(S_i, J_{ij} S_j)\beta/2] \tag{31}$$

which proves lemma 1. □

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